# ON THE BOHNENBLUST-HILLE INEQUALITY AND A VARIANT OF LITTLEWOOD'S 4/3 INEQUALITY

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ABSTRACT. The search for sharp constants for inequalities of the type Littlewood's 4/3 and Bohnenblust-Hille, besides its pure mathematical interest, has shown unexpected applications in many different fields, such as Analytic Number Theory, Quantum Information Theory, or (for instance) in deep results on the $n$-dimensional Bohr radius. The recent estimates obtained for the multilinear Bohnenblust-Hille inequality (in the case of real scalars) have been recently used, as a crucial step, by A. Montanaro in order to solve problems in the theory of quantum XOR games. Here, among other results, we obtain new upper bounds for the Bohnenblust-Hille constants in the case of complex scalars. For bilinear forms, we obtain the optimal constants of variants of Littlewood's 4/3 inequality (in the case of real scalars) when the exponent 4/3 is replaced by any $r \geq \frac{4}{3}$. As a consequence of our estimates we show that the optimal constants for the real case are always strictly greater than the constants for the complex case.

## 1. INTRODUCTION

Let $\mathbb{K}$ stand for either $\mathbb{R}$ or $\mathbb{C}$. Littlewood's 4/3 inequality [16] (see also [12]) asserts that there is a constant $L_{\mathbb{K}} \geq 1$ such that

$$\left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq L_{\mathbb{K}} \|U\|$$

for every bilinear form $U : \ell_\infty^N \times \ell_\infty^N \to \mathbb{K}$ and every positive integer $N$. It is well known that the exponent 4/3 is optimal and it was recently shown in [11] that the constant $L_{\mathbb{R}} = \sqrt{2}$ is also optimal. For complex scalars we just know that $L_{\mathbb{C}} \leq 2/\sqrt{\pi}$.

However, if we replace 4/3 by $r > 4/3$, it is not difficult to prove that the optimal constant $L_{\mathbb{K},r}$ satisfying

$$(1.1) \qquad \left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^{r} \right)^{\frac{1}{r}} \leq L_{\mathbb{K},r} \|U\|$$

is smaller than $\sqrt{2}$ (real case) and $2/\sqrt{\pi}$ (complex case). In this note, among other results, we obtain the optimal constants $L_{\mathbb{R},r}$ for all $r \geq \frac{4}{3}$; in fact, we prove that

$$L_{\mathbb{R},r} = \begin{cases} 2^{\frac{2-r}{r}} & \text{for } r \in \left[\frac{4}{3}, 2\right) \\ 1 & \text{for } r \geq 2. \end{cases}$$

As a consequence of our estimates we show that

$$L_{\mathbb{R},r} > L_{\mathbb{C},r}$$

for all $r \in \left[\frac{4}{3}, 2\right)$.

Bohnenblust and Hille's inequality [4] is an improvement of Littlewood's $4/3$ inequality, generalized to multilinear forms (see also [6, 8, 7] for recent approaches): for every positive integer $m$ there is a constant $C_m \geq 1$ so that

$$\left( \sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1}, \ldots, e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z_1,\ldots,z_m \in \mathbb{D}^N} |U(z_1, ..., z_m)|$$

for every $m$-linear form $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$ and every positive integer $N$ (for polynomial versions of the Bohnenblust-Hille inequality we refer to [6]). The first upper estimate for $C_m$ is $m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$, which was further improved to $2^{\frac{m-1}{2}}$ in [15], to $\left( \frac{2}{\sqrt{\pi}} \right)^{m-1}$ in [23] and, recently even better constants, with optimal asymptotic behavior, were obtained in [22, 10] (for related results see [11, 17, 20]).

The original motivation of the Bohnenblust-Hille inequality rests on the famous Bohr's absolute convergence problem, which consists in determining the maximal width $T$ of the vertical strip in which a Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ converges uniformly but not absolutely. The Bohnenblust-Hille inequality is a crucial tool to give a final solution to Bohr's problem: $T = 1/2$.

In Section 2 we improve the best known constants for the complex Bohnenblust–Hille inequality. Besides the intrinsic mathematical interest of finding sharper constants for famous inequalities, the search for better constants in Bohnenblust–Hille type inequalities has a long history motivated by concrete goals. As an illustration we recall that, in 2011, by proving that the polynomial Bohnenblust–Hille inequality is hypercontractive, A. Defant, L. Frerick, J. Ortega-Cerdá, M. Ounaïes and K. Seip obtained, as consequence, several new results related to the study of Dirichlet series. For instance, they obtain an ultimate generalization of a result by H. P. Boas and D. Khavinson [3] on the $n$-dimensional Bohr radius. As we already mentioned in the *Abstract*, one of the most recent applications of the Bohnenblust-Hille inequality resides in the field of Quantum Information Theory, since the exact growth of $C_m$ is related to a conjecture of Aaronson and Ambainis [1] about classical simulations of quantum query algorithms (see, also, [14]). We also mention [18] for applications of the estimates from [22] to Quantum Information Theory.

## 2. The role of Steinhaus variables. Improving the constants in the Bohnenblust–Hille inequality

Let $\varepsilon_1, \ldots, \varepsilon_n$ be a sequence of independent random variables on some probability space $(\Omega, \Sigma, P)$, having uniform (with respect to the Lebesgue measure) distribution on the complex unit-circle

$$\{z \in \mathbb{C} : |z| = 1\}.$$

These are the so-called Steinhaus random variables. The usefulness of Steinhaus random variables in the proof of the Bohnenblust–Hille inequality seems to have been first observed by H. Queffélec [23]. In our present approach we change the proof presented in [7, 22] by replacing the usual Rademacher functions by Steinhaus variables.

The first result allowing us to improve the constants of the Bohnenblust–Hille inequality is a technical inequality (Theorem 2) which is a version (now for of Steinhaus variables) of a similar result presented in [7, 22] for Rademacher functions. The crucial point in our argument is that the constants which arise in Theorem 2 are derived from the constants that appear in the Khinchine inequality for Steinhaus variables and, as we shall see at the end of the paper, this procedure generates sharper constants for the Bohnenblust–Hille inequality.

Let us recall Khinchine's inequality (for Steinhaus variables) and other useful result:

**Theorem 1** (Khinchine's inequality). *For every $0 < p < \infty$, there exist constants $\widetilde{A_p}$ and $\widetilde{B_p}$ such that*

$$(2.1) \qquad \widetilde{A_p} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n=1}^{N} a_n \varepsilon_n \right\|_p \leq \widetilde{B_p} \left( \sum_{n=1}^{N} |a_n|^2 \right)^{\frac{1}{2}}$$

*for every positive integer $N$ and scalars $a_1, \ldots, a_N$.*

From [21, p. 151] we know that

$$(2.2) \qquad\qquad\qquad A_p \leq \widetilde{A_p}$$

for all $p$ (here $A_p$ denotes the constants that appear in the place of $\widetilde{A_p}$ in Khinchine's inequality for Rademacher functions). For example, when $p = 1$ it is well known that $A_p = \frac{1}{\sqrt{2}} \approx 0.707$ and $\widetilde{A_p} = \frac{\sqrt{\pi}}{2} \approx 0.886$.

For details on the Khinchine inequalities we refer to [9, Theorem 1.10] for the case of Rademacher functions and to [2, Section 2] for more general cases, including the case of Steinhaus variables.

The following result, crucial for the proof of the Bohnenblust–Hille inequality, has essentially the same proof of its analogous for Rademacher functions (see [7, 22]).

**Theorem 2.** *Let $1 \leq r \leq 2$, and let $(y_{i_1,\ldots,i_m})_{i_1,\ldots,i_m=1}^{N}$ be a matrix in $\mathbb{C}$. Then*

$$\left( \sum_{i_1,\ldots,i_m=1}^{N} |y_{i_1,\ldots,i_m}|^2 \right)^{1/2} \leq \left( \widetilde{A_r} \right)^{-m} \left\| \sum_{i_1,\ldots,i_m=1}^{N} \varepsilon_{i_1} \ldots \varepsilon_{i_m} y_{i_1 \ldots i_m} \right\|_r.$$

In view of (2.2) we conclude that the constants $\left( \widetilde{A_r} \right)^{-m}$ are not greater than the constants from its analogous for Rademacher functions and for this reason we shall have better estimates for the constants in the Bohnenblust–Hille inequality.

The proof of the Bohnenblust–Hille inequality is (replacing the Rademacher functions for Steinhaus variables) the same proof as that from [22]. The difference in the constants is a consequence from the new constants from the Khinchine inequality for Steinhaus variables.

**Theorem 3.** *If $m \geq 1$, then*

$$\left( \sum_{i_1,\ldots,i_m=1}^{N} \left| U(e_{i_1},\ldots,e_{i_m}) \right|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z_1,\ldots,z_m \in \mathbb{D}^N} |U(z_1,\ldots,z_m)|$$

*for every $m$-linear form $U : \ell_\infty^N \times \cdots \times \ell_\infty^N \to \mathbb{C}$ and every positive integer $N$, with*

$$C_1 = 1,$$

$$C_m = \frac{C_{m/2}}{\left(\widetilde{A_{\frac{2m}{m+2}}}\right)^{m/2}}$$

*for m even and*

$$C_m = \left(\frac{C_{\frac{m-1}{2}}}{\left(\widetilde{A_{\frac{2m-2}{m+1}}}\right)^{\frac{m+1}{2}}}\right)^{\frac{m-1}{2m}} \left(\frac{C_{\frac{m+1}{2}}}{\left(\widetilde{A_{\frac{2m+2}{m+3}}}\right)^{\frac{m-1}{2}}}\right)^{\frac{m+1}{2m}}.$$

*for m odd.*

As mentioned before, from [21] we know that $A_p \leq \widetilde{A_p}$ for all $p$ and we easily conclude that the constants from Theorem 3 are better than the constants from the similar result from [22]. For $p = 1$, J. Sawa [24] has shown that the best value for $\widetilde{A_p}$ is

$$\widetilde{A_1} = \frac{\sqrt{\pi}}{2}.$$

Since $C_1 = 1$, we have

$$C_2 = \frac{2}{\sqrt{\pi}},$$

as obtained previously by Queffélec [23]. The evaluation of the precise values for $C_m$ rests on the evaluation of precise values for $\widetilde{A_p}$ with

$$p \in \left\{\frac{2m}{m+2} : m \geq 2\right\} \cup \left\{\frac{2m-2}{m+1} : m \geq 3\right\} \cup \left\{\frac{2m+2}{m+3} : m \geq 3\right\} \subset [1, 2).$$

As an "*Added in proof*" in the same paper [24], J. Sawa asserts that the sharpest constants for the parameter $p$, with $p_0 < p < 2$ and $p_0 \in (0, 2)$ defined as the unique root of the equation

$$2^{p/2} \cdot \Gamma\left(\frac{p+1}{2}\right) = \sqrt{\pi}\left(\Gamma\left(\frac{p+2}{2}\right)\right)^2$$

are

(2.3) $$\widetilde{A_p} = \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{\frac{1}{p}}.$$

A 4-digit approximation provides $p_0 \approx 0.4756$. However, Sawa presented no proof for his claim. But, fortunately, for $p \geq 1$ H. König proved that (2.3) is, in fact, the precise value of $\widetilde{A_p}$ (see [2, Section 2] and references therein). Using these values for $\widetilde{A_p}$ we construct the following table, where one can check the different estimates for $C_m$ that have been obtained so far.

| $m$ | new constants | [22] (2012) | $\left(\frac{2}{\sqrt{\pi}}\right)^{m-1}$ ([23, 8], 1995) | $2^{\frac{m-1}{2}}$ ([15],1978) | $m^{\frac{m+1}{2m}}2^{\frac{m-1}{2}}$ ([4],1931) |
|---|---|---|---|---|---|
| 2 | $\approx 1.1284$ | $-$ | $\approx 1.1284$ | $\approx 1.414$ | $\approx 2.378$ |
| 3 | $\approx 1.2364$ | $-$ | $\approx 1.273$ | $2$ | $\approx 4.160$ |
| 4 | $\approx 1.3155$ | $-$ | $\approx 1.437$ | $\approx 2.828$ | $\approx 6.726$ |
| 5 | $\approx 1.3982$ | $-$ | $\approx 1.621$ | $4$ | $\approx 10.506$ |
| 6 | $\approx 1.4637$ | $-$ | $\approx 1.829$ | $\approx 5.657$ | $\approx 16.088$ |
| 7 | $\approx 1.5224$ | $\approx 1.929$ | $\approx 2.064$ | $8$ | $\approx 24.322$ |
| 8 | $\approx 1.5714$ | $\approx 2.031$ | $\approx 2.329$ | $\approx 11.313$ | $\approx 36.442$ |
| 9 | $\approx 1.6298$ | $\approx 2.172$ | $\approx 2.628$ | $16$ | $\approx 54.232$ |
| 10 | $\approx 1.6800$ | $\approx 2.292$ | $\approx 2.965$ | $\approx 22.627$ | $\approx 80.283$ |
| 11 | $\approx 1.7256$ | $\approx 2.449$ | $\approx 3.346$ | $32$ | $\approx 118.354$ |
| 12 | $\approx 1.7659$ | $\approx 2.587$ | $\approx 3.775$ | $\approx 45.425$ | $\approx 173.869$ |
| 13 | $\approx 1.8061$ | $\approx 2.662$ | $\approx 4.260$ | $64$ | $\approx 254.680$ |
| 14 | $\approx 1.8422$ | $\approx 2.728$ | $\approx 4.807$ | $\approx 90.509$ | $\approx 372.128$ |
| 15 | $\approx 1.8757$ | $\approx 2.805$ | $\approx 5.425$ | $128$ | $\approx 542.574$ |
| 16 | $\approx 1.9060$ | $\approx 2.873$ | $\approx 6.121$ | $\approx 181.019$ | $\approx 789.612$ |
| 100 | $\approx 3.2968$ | $\approx 7.603$ | $\approx 1.55973 \cdot 10^3$ | $\approx 7.96131459 \cdot 10^{14}$ | $\approx 8.14675743 \cdot 10^{15}$ |

2.1. **Remarks on the optimal constants satisfying the complex Bohnenblust–Hille inequality.** Let $(K_n)_{n=1}^\infty$ be the sequence of the best constants satisfying the complex Bohnenblust–Hille inequality. In [19] it was recently shown that $(K_n)_{n=1}^\infty$ does not have a polynomial growth and, besides, if

$$K_n \sim n^q,$$

then

$$0 \le q \le \log_2\left(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}}\right) \approx 0.52632,$$

where $\gamma$ denotes the famous Euler-Mascheroni constant

$$\gamma = \lim_{m\to\infty}\left(\sum_{k=1}^m \frac{1}{k} - \log m\right) \approx 0.57721.$$

Since $\widetilde{A_p} = \left(\Gamma\left(\frac{p+2}{2}\right)\right)^{\frac{1}{p}}$, we have

$$\widetilde{A_{\frac{2m}{m+2}}} = \left(\Gamma\left(\frac{\frac{2m}{m+2}+2}{2}\right)\right)^{\frac{m+2}{2m}}$$

and

$$\frac{C_m}{C_{m/2}} = \left(\Gamma\left(\frac{\frac{2m}{m+2}+2}{2}\right)\right)^{\frac{-m-2}{4}}.$$

Using some basic properties of the Gamma function we can prove that

$$\lim_{m\to\infty}\frac{C_m}{C_{m/2}} = \lim_{m\to\infty}\left(\Gamma\left(\frac{\frac{2m}{m+2}+2}{2}\right)\right)^{\frac{-m-2}{4}} = e^{\frac{1}{2}-\frac{1}{2}\gamma} \approx 1.23539,$$

and following the arguments from the Dichotomy Theorem (see [19]) we conclude that if $K_n \sim n^q$, then

$$0 < q \le \log_2\left(e^{\frac{1}{2}-\frac{1}{2}\gamma}\right) \approx 0.30497,$$

as we wished. We mention that a more complete results on this line were recently proved in [20].

## 3. THE VARIANTS OF LITTLEWOOD'S 4/3 INEQUALITY

3.1. **Real case.** As mentioned in the Introduction, if we replace $4/3$ by $r > 4/3$, then the optimal constant $L_{\mathbb{K},r}$ satisfying

$$(3.1) \qquad \left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^r \right)^{\frac{1}{r}} \leq L_{\mathbb{K},r} \|U\|$$

is smaller than $\sqrt{2}$. Our main goal is to find the optimal values of $L_{\mathbb{K},r}$ for all $r \geq 4/3 :$

**Theorem 4.** *The optimal constant $L_{\mathbb{R},r}$ satisfying (1.1) is*

$$L_{\mathbb{R},r} = \left\{ \begin{array}{c} 2^{\frac{2-r}{r}} \text{ for } r \in \left[\frac{4}{3}, 2\right) \\ 1 \text{ for } r \geq 2. \end{array} \right.$$

*Proof.* The case $r \geq 2$ is quite simple. In fact, one can use that the real scalar field has cotype 2 and its cotype constant is 1. Hence

$$(3.2) \qquad \left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} \leq \|U\|$$

for all $N$ and all bilinear forms $U : \ell_\infty^N \times \ell_\infty^N \to \mathbb{R}$. Using (3.2) and the monotonicity of the $\ell_r$ norms we conclude that $L_{\mathbb{R},r} \leq 1$. On the other hand, using $U_0(x,y) = x_1 y_1$ in (1.1) we conclude that $L_{\mathbb{R},r} \geq 1$. Now we deal with the case $r \in [\frac{4}{3}, 2)$. Using a simple interpolation argument we can show that if $\theta \in (0,1)$ is so that

$$\frac{1}{r} = \frac{\theta}{4/3} + \frac{1-\theta}{2},$$

then

$$\left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^r \right)^{\frac{1}{r}} \leq \left( \left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^{4/3} \right)^{\frac{3}{4}} \right)^{\theta} \left( \left( \sum_{i,j=1}^{N} |U(e_i, e_j)|^2 \right)^{\frac{1}{2}} \right)^{1-\theta}$$

$$\leq \left( \sqrt{2} \right)^{\theta} \|U\|$$

$$= 2^{\frac{2-r}{r}} \|U\|.$$

On the other hand, by considering

$$(3.3) \qquad U_1(x,y) = x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2$$

we have $\|U_1\| = 2$ and thus

$$L_{\mathbb{R},r} \geq \frac{4^{\frac{1}{r}}}{\|U_1\|} = 2^{\frac{2-r}{r}}.$$

$\square$

3.2. **Complex case.** We will show that

$$L_{\mathbb{C},r} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{4-2r}{r}} \quad \text{for all } r \in \left[\frac{4}{3}, 2\right)$$

and for $r \geq 2$ it is straightforward that $L_{\mathbb{C},r} = 1$. However, we do not prove that our estimates are optimal for $r \in \left[\frac{4}{3}, 2\right)$.

For the case of complex scalars the more accurate known estimate for the constant in the Littlewood's 4/3 theorem is

$$L_{\mathbb{C},\frac{4}{3}} \leq \frac{2}{\sqrt{\pi}}.$$

The same interpolation argument used in the case of real scalars can be used to show that if $\theta \in (0,1)$ is so that

$$\frac{1}{r} = \frac{\theta}{4/3} + \frac{1-\theta}{2},$$

then

$$\left(\sum_{i,j=1}^{N} |U(e_i, e_j)|^r\right)^{\frac{1}{r}} \leq \left(\left(\sum_{i,j=1}^{N} |U(e_i, e_j)|^{4/3}\right)^{\frac{3}{4}}\right)^{\theta} \left(\left(\sum_{i,j=1}^{N} |U(e_i, e_j)|^2\right)^{\frac{1}{2}}\right)^{1-\theta}$$

$$\leq \left(\frac{2}{\sqrt{\pi}}\right)^{\theta} \|U\|$$

$$= \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{4-2r}{r}} \|U\|.$$

For $r \geq 2$, it is simple to show that the exact value is $L_{\mathbb{C},r} = 1$. However our technique to provide lower estimates for $L_{\mathbb{R},r}$ seems useless for the complex case. The following table is illustrative:

| $r$ | $L_{\mathbb{C},r} \geq$ | $L_{\mathbb{C},r} \leq$ |
|---|---|---|
| | 1 | $\left(\frac{2}{\sqrt{\pi}}\right)^{\frac{4-2r}{r}}$ |
| 4/3 | 1 | $2/\sqrt{\pi} \approx 1.128380$ |
| 1.93 | 1 | 1.0088 |
| 1.95 | 1 | 1.0062 |
| 1.99 | 1 | 1.0012 |
| $\geq 2$ | 1 | 1 |

Since $L_{\mathbb{R},r} = 2^{\frac{2-r}{r}}$ and $L_{\mathbb{C},r} \leq \left(\frac{2}{\sqrt{\pi}}\right)^{\frac{4-2r}{r}}$ for all $r \in \left[\frac{4}{3}, 2\right)$, it follows that

$$(3.4) \qquad\qquad\qquad L_{\mathbb{R},r} > L_{\mathbb{C},r}$$

for all nontrivial cases, i.e., whenever $r \in \left[\frac{4}{3}, 2\right)$.

3.3. **Some remarks.** We do not know if our estimates for complex scalars are optimal. We now stress that a different technique, although quite effective for estimates of the Bohnenblust–Hille inequality, provides worse results. This approach is based on recent arguments from [7, 11, 22]). For the sake of completeness, let us recall two useful results:

**Theorem 5** (Khinchine's inequality)**.** *For all $0 < p < \infty$, there exist constants $A_p$ and $B_p$ such that*

$$(3.5) \qquad A_p \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \left| \sum_{n=1}^N a_n r_n(t) \right|^p dt \right)^{\frac{1}{p}} \leq B_p \left( \sum_{n=1}^N |a_n|^2 \right)^{\frac{1}{2}}$$

*for every positive integer $N$ and scalars $a_1, \ldots, a_n$ ($r_n$ denotes the n-th Rademacher function).*

For $p > p_0$ with $1 < p_0 < 2$ defined by

$$\Gamma \left( \frac{p_0 + 1}{2} \right) = \frac{\sqrt{\pi}}{2},$$

a result due to U. Haagerup ([13]) asserts that

$$(3.6) \qquad A_p := \sqrt{2} \left( \frac{\Gamma((p+1)/2)}{\sqrt{\pi}} \right)^{1/p}$$

are the best constants satisfying (3.5); for $p \leq p_0$ the best values are

$$(3.7) \qquad A_p = 2^{\frac{1}{2} - \frac{1}{p}}.$$

**Theorem 6** (Blei, Defant et al., [7, Lemma 3.1])**.** *Let $A$ and $B$ be two finite non-void index sets, and $(a_{ij})_{(i,j) \in A \times B}$ a scalar matrix with positive entries, and denote its columns by $\alpha_j = (a_{ij})_{i \in A}$ and its rows by $\beta_i = (a_{ij})_{j \in B}$. Then, for $q, s_1, s_2 \geq 1$ with $q > \max(s_1, s_2)$ we have*

$$\left( \sum_{(i,j) \in A \times B} a_{ij}^{w(s_1, s_2)} \right)^{\frac{1}{w(s_1, s_2)}} \leq \left( \sum_{i \in A} \|\beta_i\|_q^{s_1} \right)^{\frac{f(s_1, s_2)}{s_1}} \left( \sum_{j \in B} \|\alpha_j\|_q^{s_2} \right)^{\frac{f(s_2, s_1)}{s_2}},$$

*with*

$$w : [1, q)^2 \to [0, \infty), \ w(x, y) := \frac{q^2(x + y) - 2qxy}{q^2 - xy},$$

$$f : [1, q)^2 \to [0, \infty), \ f(x, y) := \frac{q^2 x - qxy}{q^2(x + y) - 2qxy}.$$

As we already know, the Khinchine inequality for Steinhaus variables has

$$\widetilde{A_p} = \left( \Gamma \left( \frac{p+2}{2} \right) \right)^{\frac{1}{p}}$$

as the optimal constant whenever $p \geq 1$ (see [2, 24]). So, using this value for $\widetilde{A_p}$ and an argument similar to the proof of the main result of [22] (which has its roots in [7]) with

$$\begin{cases} s_1 = s_2 = \frac{2r}{4-r}, \\ \qquad q = 2 \end{cases}$$

in Theorem 6, we have

$$\begin{cases} w(s_1, s_2) = r, \\ f(s_1, s_2) = 1/2. \end{cases}$$

and we obtain

$$L_{\mathbb{C},r} \le \left( \left( \Gamma\left( \frac{\frac{2r}{4-r}+2}{2} \right) \right)^{\frac{4-r}{2r}} \right)^{-1} = \left( \Gamma\left( \frac{4}{4-r} \right) \right)^{\frac{r-4}{2r}}$$

for all $r \in \left[\frac{4}{3}, 2\right)$. But a direct inspection shows that

$$\left( \Gamma\left( \frac{4}{4-r} \right) \right)^{\frac{r-4}{2r}} > \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{4-2r}{r}}$$

for all $r \in \left(\frac{4}{3}, 2\right)$ and thus the estimates of Subsection 3.2 are more precise.

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